APPLICATIONS OF NETWORK FLOWS IN LAGRANGIAN RELAXATION

By

PIYUSH SHANKAR GARG

DEPARTMENT OF INDUSTRIAL & MANAGEMENT ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

MARCH, 1993

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A Thesis Submitted
in Partial Fulfilment of the Requirements
for the Degree of
MASTER OF TECHNOLOGY

By

PIYUSH SHANKAR GARG

to the
DEPARTMENT OF INDUSTRIAL & MANAGEMENT ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY KANPUR

MARCH, 1993

CERTIFICATE

This is to certify that the present work entitled **"Applications of Network Flows in Lagrangian Relaxation"** has been carried out by Mr. Piyush Shankar Garg under my supervison and that it has not been submitted elsewhere for a degree.

March 12, 1993

Dr. R. K. Ahuja
Associate Professor
Dept of Industrial &
Management Engineering
Indian Institute of Technology
Kanpur-208016, INDIA

Dedicated to my teacher

Late Shri Ramsharan Sharma 'Bhaisahab'

and

to my parents and sister.

ACKNOWLEDGEMENTS

I find myself at a loss for words, to express my gratitude to the person who made my stay and study here so memorable and a thing to cherish throughout my life. Dr R K Ahuja is to me more than a mere Thesis Supervisor or mentor. I feel deeply indebted and express my heartfelt thanks to him for his enthusiastic guidance, inspiring support through the course of this thesis work.

I express my gratitude to all the faculty members of IME Department specially to Dr Kripa Shanker, Dr J L Batra and Dr S Sadagopan for making my stay at IIT memorable.

I express my sincere thanks to Rajan, Raj, Alok, Rakesh, Sahay and Shah, whom I have come to look on as brothers than as friends for the warmth of their companionship. Special thanks are also due to Shiva, Neeraj, Jhaji, Mukul, Sreekanth, Mishra, Saibal, Mathews, Bhaji, Raman, NJM Reddy , Swami for their help and support.

I am also grateful to Shri G L Misra and Shri J K Misra for excellent typing of the manuscript.

I also place on record the help rendered by IME family.

12th March, 1993 (Piyush Shankar Garg)

ABSTRACT

Network flows is an important subfield of Operations Research and is well-known for its diverse applications and highly efficient algorithms. Applications of network flow models have been identified in a variety of situations including applied mathematics, communication systems, defense, manufacturing and production, scheduling, and public policy. In the recent past, researchers have made efforts to identify, compile and summarize those applications which can be transformed into fundamental network flow problems such as shortest paths, maximum flows, minimum cost flows, assignments and minimum spanning trees, etc. This thesis addresses a similar issue and attempts to identify the applications of network flow models in Lagrangian Relaxation.

Lagrangian relaxation is an important technique in combinatorial optimization and is extensively used in designing branch and bound algorithms for hard (i.e., NP-complete) problems. The relaxed problems obtained by applying the Lagrangian relaxation technique are often network flow problems and, therefore, can be very efficiently solved by network flow algorithms.

In this work, we make an attempt to identify and summarize such applications whose lagrangian relaxation results in network flow subproblems. We report over 40 applications which have been selected through an extensive research of the available literature. These also include applications which we have formulated ourselves. We hope that the understanding of the models described here will allow researchers to identify more and more hard optimization problems whose lower bounds can be obtained more efficiently by using network flow algorithms in Lagrangian relaxation.

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CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION

Network flows is an important area in linear programming and is known for its novel theoretical properties. For most of the network flow problems, very efficient algorithms have been developed by exploiting the structure of the constraint matrix. However, many real life and practical problems can be viewed as easy network flow problems complicated by a relatively small set of side constraints. These problems can not be modeled as pure network flow problems and are difficult to solve as direct algorithms are not available for such constrained network flow problems.

Lagrangian relaxation is a flexible solution strategy that permits to exploit underlying structure in any optimization problem by relaxing complicated constraints. This approach permits to "pull apart" constrained network flow problems by removing complicating non-network constraints and instead place them in the objective function with associated Lagrangian multipliers. Dualizing these complicating side constraints using the Lagrangian relaxation technique produces Lagrangian subproblem (or subproblems) that will be a pure network problem in nature. This Lagrangian subproblem can be solved as a pure network flow problem and its optimal value will be a lower bound (for minimization problem) on the optimal value of the original constrained network flow problem. Many practitioners identified various applications of such hard combinatorial optimization problems which are having two sets of constraints, one set of "network" constraints and the other of non-network side constraints. These

applications are few in number and appear to be scattered in the literature.

Though Fisher [1981] and Ahuja, Orlin and Magnanti [1993] gave a compilation of few such applications, yet there is no single paper, or any other reference that summarizes all such real life applications at one place. This report fills precisely this need by describing, summarizing or simply referencing over 40 such combinatorial optimization problems in which by relaxing complicating non-network constraints, we can get pure network flow Lagrangian subproblems. Furthermore, we also tried to identify and add some applications of similar nature from our side in this report. The network flow problems, which arise as Lagrangian subproblems in such relaxed hard combinatorial optimization problems, are:

(i) The shortest path problem.

(ii) The minimum spanning tree problem.

(iii) The minimum cost flow problem.

(iv) The assignment problem.

(v) Convex cost flow problem.

(vi) Generalized flow problem.

(vii) Multicommodity flow problem.

Identifying such constrained network flow application or combinatorial optimization problems with two sets of constraints, proved to be a difficult task. Since there existed no reference on such applications, we almost had to survey the entire combinatorial optimization/network flows area.

In the first phase, we went over the papers cited in the Integer Programming Bibliographies (Kasting [1976], Hausman [1978] and Von Randow [1982, 1985]) under the subjects lying within relaxation, network flows and

integer programming. We also went through the international abstracts of operation research in above areas. We then scanned these papers to identify such applications. We also scanned the papers given in the reference list of above identified papers. In the second phase, we went through many journals in management science, operations research, mathematics and communications where such articles have appeared in the past. We also thought some additional applications from our side in this phase.

During our research, the applications we found, could be categorized into two broad classes:

(i) Applications whose Lagrangian relaxation resulted into one single pure network flow problem.

(ii) Applications whose relaxation resulted into more than one, different type of pure network flow subproblems.

We have included in our report most of the such applications that were interesting, simple to describe and throw insight on modeling and relaxing techniques. We have kept the readability and the importance of an application in mind while deciding its inclusion in the text. There were another category of applications that were nice but either too related to the applications that we described completely or rather simple to be discussed fully. For such applications, we only describe the problem. To summarize, the applications we identified are covered in this report in the following three manners:

(i) The problem is described and its formulation is shown. By using Lagrangian relaxation, resulting pure network flow subproblem (or subproblems) is shown for such application.

(ii) The problem is described and its equivalence with previously

described problem has been proved.

(iii) The problem is simply stated with no description of the model given, because of the complexity of the problem.

This report serves the following purposes. For the first time it collects together all such real life applications which are network flow problems with additional side constraints. These application will throw insight on the use of Lagrangian relaxation technique to solve such hard problems. These applications are of great use to teachers who may include some of these in their courses in combinatorial optimization, Operations Research and Network Flows. We believe that the understanding of the models described here will allow the researchers to bring more and more hard optimization problems those can be solved as network flow problems using Lagrangian relaxation.

1.2 NOTATION AND DEFINITION

We now collect together several basic definitions and describe some notation. We consider a directed graph $\mathbf{G = (N,A)}$ consisting of a set, N, of nodes, and a set, A, of arcs whose elements are ordered pairs of distinct nodes. A directed network is a directed graph with numerical values attached to its nodes and/or arcs. We let $n = |N|$ and $m = |A|$. We associate with each arc $(i,j) \in \mathbf{A}$, a cost c_{ij} and a capacity $u_{ij} \geq 0$. Frequently, we distinguish two special nodes in a graph: the source s and sink t.

An arc (i,j) has two end points, i and j. We refer to node i as the tail and node j as the head of arc (i,j), and the arc (i,j) is incident to nodes i and j. The arc (i,j) is an outgoing arc of node i and an incoming arc of node j. The arc adjacency list of node i, A(i), is defined as the set of arcs emanating from node i, i.e., $\mathbf{A(i) = \{(i,j) \in A : j \in N\}}$. The

node-arc incidence matrix representation, or simply the incidence matrix representation, represents a network as the constraint matrix of the problem. This representation stores the network as an $n \times m$ matrix $\mathbb{N}$ which contains one row for each node of the network and one column for each arc. The column corresponding to arc (i,j) has only two nonzero elements: it has a +1 in the row corresponding to row i and a -1 in the row corresponding to row j. The degree of a node is the number of incoming and outgoing arcs incident to that node.

A directed path in $G = (N,A)$ is a sequence of distinct nodes and arcs i_1, (i_1, i_2), $i_2, (i_2, i_3), i_3, \ldots, (i_{r-1}, i_r), i_r$ satisfying the property that $(i_k, i_{k+1}) \in A$ for each $k = 1, \ldots, r-1$. An undirected path is defined similarly except that for any two consecutive nodes i_k and i_{k+1} on the path, the path contains either the arc (i_k, i_{k+1}) or the arc (i_{k+1}, i_k). A directed cycle is a directed path together with the arc (i_r, i_1) and an undirected cycle is an undirected path together with the arc (i_r, i_1) or (i_1, i_r).

We shall often use the terminology *path* to designate either a directed or an undirected path, whichever is appropriate from context. For simplicity of notation, we shall often refer to a path as a sequence of nodes $i_1 - i_2 - \ldots - i_k$ when its arcs are apparent from the problem context. Alternatively, we shall sometimes refer to a path as a set of (sequence of) arcs without any mention of the nodes. We shall use similar conventions for representing cycles.

A graph $G = (N,A)$ is called a bipartite graph if its node set N can be partitioned into two subsets N_1 and N_2 so that for each arc (i,j) in A, $i \in N_1$ and $j \in N_2$.

A graph $G' = (N',A')$ is a subgraph of $G = (N,A)$ if $N' \subseteq N$ and $A' \subseteq A$.

A graph $G' = (N', A')$ is a spanning subgraph of $G = (N, A)$ if $N' = N$ and $A' \subseteq A$.

Two nodes i and j are said to be connected if the graph contains at least one undirected path from i to j. A graph is said to be connected if all pairs of its nodes are connected; otherwise, it is disconnected. The connected subgraphs of a graph are called its components. We always assume that the graph G is connected and hence $m \geq n-1$.

1.3 NETWORK FLOW PROBLEMS

We shall now briefly describe the network flow problems which arise as Lagrangian subproblems in the relaxed problems considered in this thesis.

Shortest Path Problem

The shortest path problem is to determine directed paths of smallest cost from a given node s to all other nodes in the network. The problem can be stated as the following linear programming problem:

$$\text{minimize} \sum_{(i,j)\in A} C_{ij} X_{ij}$$

subject to

$$\sum_{\{j:(i,j)\in A\}} X_{ij} - \sum_{\{j:(j,i)\in A\}} X_{ji} = \begin{cases} (n-1) & \text{for } i = s, \\ -1 & \text{for all } i \in N - \{s\}, \end{cases}$$

$$X_{ij} \geq 0, \text{ for all } (i,j) \in A.$$

Some of the simplest applications of the shortest path problem are to determine a path between two specified nodes of a network that has a minimum length, or a path that takes least time to traverse, or a path that has the maximum reliability.

Assignment Problem

The data of the assignment problem consists of a set, N_1, say of persons and a set N_2, say of objects, satisfying $|N_1| \equiv |N_2|$, a collection of node pairs $A \subseteq N_1 \times N_2$ representing possible person-to-object assignments, and a cost C_{ij} associated with each element (i,j) in A. The objective is to assign each person to exactly one object in a way that minimizes the total cost of assignment. This problem can be stated as the following linear program:

$$\text{minimize} \quad \sum_{(i,j)\in A} C_{ij} X_{ij}$$

subject to

$$\sum_{\{j:(i,j)\in A\}} X_{ij} = 1 \text{ for all } i \in N_1,$$

$$\sum_{\{j:(i,j)\in A\}} X_{ij} = 1 \text{ for all } j \in N_2,$$

$$X_{ij} \geq 0, \text{ for all } (i,j) \in A.$$

Sometimes, it may be unbalanced problem that can be transformed to a balanced assignment problem by introducing some dummy nodes. Sometimes, we have a utility u_{ij} assigned with each arc $(i,j) \in A$ and our objective is to obtain an assignment that maximizes the total utility $\sum_{(i,j)\in A} u_{ij} X_{ij}$. We can transform this problem to the previous version by defining $C_{ij} = - u_{ij}$ and minimizing the objective $\Sigma_{(i,j)\in A} u_{ij} X_{ij}$.

The applications of the assignment problem include assigning people to projects, jobs to machines, tenants to apartments, swimmer to events in a swimming meet, and medical school graduates to available internship.

Minimum Spanning Tree Problem

A spanning tree is a connected acyclic graph that touches all the nodes of an undirected network. The cost of a spanning tree is the sum of the costs of its arcs. In minimum spanning tree problem, we wish to identify a spanning tree of minimum cost.

Minimum Cost Flow Problem

We wish to determine a least cost shipment of a commodity through a network that will satisfy demands at certain nodes from available supplies at other nodes. The problem can be stated as the following linear programming problem:

$$\text{minimize} \sum_{(i,j)\in A} C_{ij} X_{ij}$$

subject to

$$\sum_{\{j:(i,j)\in A\}} X_{ij} - \sum_{\{j:(j,i)\in A\}} X_{ji} = b(i) ,$$

$$l_{ij} \le X_{ij} \le u_{ij} .$$

In this problem, C_{ij} represents the cost per unit flow on arc (i,j), l_{ij} represents the lower bound on the arc flow and u_{ij} represents upper bound on arc flow. The number b(i) represents the supply/demand of a node $i \in N$. If $b(i) > 0$, then node i is a supply node; if $b(i) < 0$ then node i is a demand node; and if $b(i) = 0$ then it is a transshipment node.

The applications of minimum cost flow problem include : the distribution of a product from manufacturing plants to warehouses, or from warehouses to retailers; the flow of raw material and intermediate goods through the various machining stations in a production line; the routing of automobiles through an urban street network; and the routing of calls

through the telephone system.

Convex Cost Flow Problem

In the minimum cost flow problem, we assume that the cost of the flow on any arc varies linearly with the amount of flow. Convex cost flow problems have a more general cost structure: the cost is a convex function of the amount of flow. The problem can be formulated as follows :

$$\text{minimize} \sum_{(i,j)\in A} C_{ij}\,(X_{ij})$$

subject to

$$\sum_{(j:(i,j)\in A)} X_{ij} - \sum_{(j:(j,i)\in A)} X_{ji} = b(i), \text{ for all } i \in N.$$

$$0 \le X_{ij} \le u_{ij}, \text{ for all } (i,j) \in A,$$

$$X_{ij} \text{ is integer for all } (i,j) \in A.$$

We define this model on a directed network $G = (N,A)$ with a capacity u_{ij} and a convex cost function $C_{ij}(X_{ij})$ associated with every arc $(i,j) \in A$. As always, we associate a number $b(i)$ with each node $i \in N$ specifying the node's supply or demand, depending upon whether $b(i) > 0$ or $b(i) < 0$.

Flow costs vary in a convex manner in numerous problem settings including (i) power loss in a electrical network due to resistance; (ii) congestion costs in a city transportation network; and (iii) expansion costs of a communication network.

Generalized Flow Problem

In the minimum cost flow problem, arc conserve flows, i.e., the flow entering an arc equals the flow leaving the arc. In generalized flow problems, arc might "consume" or "generate" flow. If X_{ij} units of flow enter an arc (i,j), then $\mu_{ij}X_{ij}$ units arrive at node j; μ_{ij} is a positive multiplier associated with the arc. If $0 < \mu_{ij} < 1$, then the arc is lossy,

and if $1 < \mu_{ij} < \infty$, then the arc is gainy. The problem can be formulated as follows :

$$\text{minimize} \sum_{(i,j)\in A} C_{ij} X_{ij}$$

subject to

$$\sum_{\{j:(i,j)\in A\}} X_{ij} - \sum_{\{j:(j,i)\in A\}} \mu_{ji} X_{ji} = b(i) \text{ for } i \in N,$$

$$0 \le X_{ij} \le u_{ij}, \ (i,j) \in A.$$

Generalized network flow problems arise in several application contexts: for example (i) power transmission through electric lines when power is lost with distance; (ii) flow of water through pipelines or canals that lose water due to seepage or evaportion; (iii) cash management scenarios in which arcs represent investment opportunities and multipliers represent appreciation or depreciation of an investment's value.

Multicommodity Flow Problem

The multicommodity problem arises when several commodities use the same underlying network. The commodities may either be differentiated by their physical characteristics or by simply their origin-destination pairs. Different commodities have different origins and destinations, and commodities have separate mass balance constraints at each node. However, the sharing of common arc capacities binds the different commodities together. In fact, in multicommodity problems we deal with the allocation of the capacity of each arc to the individual commodities in a way to minimize the overall flow cost.

Let X^k_{ij} denote the flow of commodity k on arc (i,j), and let C^k_{ij} denote per unit cost of transportation of commodity k on arc (i,j). The problem can be formulated as follows :

$$\text{minimize} \sum_{k=1}^{K} \sum_{(i,j)\in A} C_{ij}^{k} X_{ij}^{k}$$

subject to

$$\sum_{\{j:(i,j)\in A\}} X_{ij}^{k} - \sum_{\{j:(j,i)\in A\}} X_{ji}^{k} = b_{i}^{k}$$

$$\text{for all } i \in N,\ k = 1,\ldots,K,$$

$$\sum_{k=1}^{K} X_{ij}^{k} = u_{ij} \qquad \text{for all } (i,j) \in A,$$

$$0 \le X_{ij}^{k} \le u_{ij}^{k} \qquad \text{for all } (i,j) \in A \text{ and } k = 1,2,\ldots,K.$$

The total flow of all the commodities on each arc (i,j) has been restricted to maximum arc capacity u_{ij}. Furthermore, individual flow bounds u_{ij}^{k} on the flow of commodity k on arc (i,j) has been imposed.

Multicommodity flow problems arise in many practical situations including (i) the transportation of passengers from different origins to different destinations within a city; (ii) the worldwide shipments of different varieties of grains from countries that produce grain to those which consume it.

1.4 OVERVIEW OF THE THESIS

In Chapter 2, we describe the Lagrangian Relaxation technique and related properties and theorems. In Chapter 3, we describe applications in which relaxed Lagrangian subproblems are shortest path problems. Chapter 4 includes applications in which relaxed Lagrangian subproblems are either undirected minimum spanning tree problems or directed minimum spanning tree problems. In Chapter 5, we describe applications whose Lagrangian relaxation result in more than one Lagrangian network flow subproblems. In addition to these, we also report the additional applications in which Lagrangian subproblems will be network flow problems

other than shortest path and minimum spanning tree problems.

We have mentioned references against each application. We have not mentioned any reference against applications which we have formulated ourselves.

CHAPTER 2

LAGRANGIAN RELAXATION

2.1 INTRODUCTION

Combinatorial optimization problems come in two varieties. There is a small number of easy problems which can be solved in time bounded by a polynomial in the input length and an all-too-large class of "Hard Problems" for which all known algorithm require exponential time in the worst case. Many hard problems can be viewed as easy problems complicated by a relatively small set of constraints.

Lagrangian relaxation is a solution strategy that dualizes side constraints and produce a Lagrangian subproblem that is easy to solve and whose optimal value is a lower bound (for minimization problem) on the optimal value of the objective function of the original problem. The Lagrangian relaxation can thus be used in place of a linear programming relaxation to provide bounds in a branch and bound algorithm.

2.2 LITERATURE SURVEY

The Lagrangian multiplier technique of nonlinear optimization dates to the 1700s and was originated by the famous mathematician Lagrange, from whom the technique is named. There were a number of forays prior to 1970 into the use of Lagrangian methods in discrete optimization, including the Everett's proposal for "generalizing" Lagrange multipliers (1963). However, the "birth" of the Lagrangian approach as it exists today occurred in 1970 when Held and Karp (1971) used a Lagrangian problem based on minimum spanning trees to devise a dramatically successful algorithm for

the travelling salesman problem. This work was not only an eye-opening successful application, but also set out many key ideas in applying the Lagrangian relaxation method to integer programming problems. Fisher (1981) and Geoffrion (1974) provide insightful surveys of Lagrangian relaxation and its uses in integer programming. Orlin and Bertsimas (1991) have developed the most efficient algorithms (in the worst case) for many classes of Lagrangian relaxation problems. Belling-Seib, Meyer and Mûller (1988) have compared Lagrangian relaxation technique for solving the constrained network flow problems with other techniques (primal simplex algorithm and dual method).

2.3 LAGRANGIAN RELAXATION TECHNIQUE

To describe the general form of the Lagrangian relaxation procedure, we begin with the following combinatorial optimization model formulated in terms of a vector X of a decision variables :

$$Z^{*} = \text{minimize } CX \qquad \textbf{(P)}$$

subject to :

$$AX \leq b, \qquad (2.2a)$$

$$X \in \mathbb{X}. \qquad (2.2b)$$

The model has a linear objective function CX and a set $AX \leq b$ of linear explicit linear constraints. The decision variables X are also constrained to lie in a given constraint set $\mathbb{X} = \{X : \mathbb{N}X = q,\ 0 \leq x \leq u\}$ might be all the feasible solutions to a network flow problem with a supply/demand vector q. We further assume that $\mathbb{X}$ is finite.

Lagrangian relaxation procedure uses the idea of relaxing explicit constraint set (2.2a) by bringing them into the objective function by associating a Lagrangian multiplier vector μ. The resulting problem will

be:

$$\text{minimize } (CX + \mu\,(AX - b) \qquad \textbf{(L)}$$

subject to $X \in \mathbb{X}$.

We can refer to the function

$$L(\mu) = \min \{CX + \mu(AX - b) : X \in \mathbb{X}\}, \qquad (2.2c)$$

as the Lagrangian function.

If the constraint set (2.2a) is less than or equal to type, then μ will be restricted to the non-negative, i.e., ($\mu \geq 0$) values; else, if it is equality constraint set ($AX = b$), then μ will be unrestricted in sign; otherwise, if constraint set (2.2a) is greater-than or equal to type ($AX \geq b$), then μ will be restricted to negative ($\mu < 0$).

Theorem 2.1 : *(Lagrangian Bounding Principle). For non-negative Lagrangian multipliers ($\mu \geq 0$), associated with constraint set (2.2a), i.e., $AX \leq b$, the value $L(\mu)$ of the Lagrangian function (2.2c) is a lower bound on the optimal objective function value Z^* of the original optimization problem (P).*

Proof : Since $AX \leq b$ for every feasible solution to (P) and for non-negative Lagrangian vector set ($\mu \geq 0$), the result will be :

$$L(\mu) \leq CX^* + \mu(AX^* - b) \leq (CX^* = Z^*).$$

We assume that X^* is an optimal solution to (P). As (AX^*-b) will be either zero or less than zero for an optimal solution X^* to (P), then for $\mu \geq 0$, $\mu\,(AX^*-b)$ will be either zero or negative. Hence $L(\mu)$ will be a lower bound on the optimal value Z^* for the original optimization problem. To get the sharpest possible bound, we would need to solve the following optimization problem :

$$L^* = \max_{\mu \geq 0} L(\mu),$$

which we refer to Lagrangian multiplier problem associated with the original optimization problem **(P)**. The Lagrangian bounding principle has the following implication.

Theorem 2.2: *(Weak Duality Theorem). The optimal objective function L^* of the Lagrange multiplier problem is always a lower bound on the optimal objective function value of the problem (P), i.e., $L^* \leq Z^*$.* ■

Theorem 2.3: *If for some choice of the Lagrangian multiplier ($\mu \geq 0$), the solution X^* of the Lagrangian relaxation (i) is feasible in the optimization problem (P), and (ii) satisfies the complementary slackness condition $\mu(AX^*-b) = 0$. Then X^* is an optimal solution of the original optimization problem (P).*

Proof: By assumption, $L(\mu) = CX^* + \mu(AX^* - b)$. Since $\mu(AX^* - b) = 0$, $L(\mu) = CX^*$ and the solution X^* is feasible to the optimization problem **(P)**, i.e., $AX^* \leq b$. Hence, X^* is an optimal solution to the optimization problem **(P)**. ■

The above mentioned properties and theorems show that certain solutions of the Lagrangian subproblems provably solve the original problem (P). From this, we arrive at two properties :

Property 2.1: Solutions to the Lagrangian subproblem (L) that are feasible, but are not provably optimal for the original problem (P).

Property 2.2: Solution to the Lagrangian subproblem (L) are not feasible to original problem (P).

In the first case (Property 2.1), branch and bound procedures to get the sharpest possible bound can be used. Fisher (1981) provided an insight to the use of Lagrangian problem in place of linear programming relaxation

to provide bounds in branch and bound algorithms. In the second case, for many applications, researchers have been able to modify "modestly" infeasible solutions so that they become feasible with only a slight degradation in the objective function value by using some heuristic methods.

Subgradient Optimization:

The following issue is related while solving the Lagrangian multiplier problem (L): How to find out the optimal multiplier value (μ^*) of the Lagrangian multiplier problem (L), i.e., we need to find the highest point of the Lagrangian multiplier function $L(\mu)$.

For our optimization model (P) defined as $\{CX : AX \le b,\ X \in \mathbb{X}\}$, we assume that the set $\mathbb{X} = \{X^1, X^2, \ldots, X^K\}$ is finite. By relaxing the constraints $AX \le b$, we obtain the Lagrangian multiplier function $L(\mu) = \min \{CX + \mu(AX-b) : X \in \mathbb{X}\}$. Thus :

$$L(\mu) \le CX^k + \mu(AX^k - b) \text{ for all } k = 1, \ldots, \mathbb{K}.$$

The best choice for μ would be an optimal solution to the dual problem :

$$\max \omega \qquad \textbf{(D)}$$

subject to :

$$\omega \le CX^k + \mu(AX^k - b), \text{ for all } k = 1, \ldots, \mathbb{K}.$$

$$\mu \ge 0.$$

The above linear program (D) can be solved by applying linear programming methodology. Dantzig-Wolfe decomposition or generalized linear programming, is an important solution strategy to solve the linear program (D). It has been discussed by Ahuja, Magnanti and Orilin (1993). However, one of the disadvantages of this approach that it requires the solution of a series of linear programs which are rather expensive computationally.

Another approach to find the optimal value of Lagrangian multipliers might be to apply gradient method to the Lagrangian function $L(\mu)$. The function $L(\mu)$ has all properties like continuity and concavity except one-differentiability. The function $[L(\mu)]$ is non-differentiable at any μ, where it has two or more solutions. For solving the (non-differentiable) Lagrangian, subgradient optimization technique is commonly used. The subgradient method is a brazen application of the gradient method in which gradients are replaced by subgradients. Let μ^o be any initial choice of the Lagrange multiplier; the subsequent values μ^k for $k = 1,2,\ldots,$ of the Lagrange multipliers can be determined as follows :

$$[\ \mu^{k+1} = \mu^k + \theta_k(AX^k - b)]^+.$$

In the above expression, X^k is any solution to Lagrangian subproblem (L) when $\mu = \mu^k$ and θ_k is the step length at the kth iteration. As the updated value of one or more components of μ may become negative, we avoid this possibility by taking only the positive value. The notation $[Y]^+$ denotes the "positive part" of the vector y; that is, the ith component of $[Y]^+$ equals the maximum of zero and Y_i.

Held, Wolfe and Crowder (1974) have discussed the computational performance and theoretical convergence properties of subgradient optimization in detail. The step size θ_k used commonly in practice is :

$$\theta_k = \frac{[UB - L(\mu^k)]}{\|(AX^k - b)\|^2}$$

In this expression, UB is an upper bound on the optimal objective function value Z^* of the problem (P) and it can be any known feasible solution to the problem (P).

Relationship to Linear Programming

Two properties are important in evaluating a relaxation : the sharpness of the bound produced and the amount of computation required to obtain these bounds. The primary use of the Lagrangian relaxation technique is to obtain lower bounds on the objective function values (discrete) optimization problems. By relaxing the integrality constraints in the original problem (P), we obtain an alternative method for generating lower bound, known as linear relaxation.

Theorem 2.3: *Integrality property : Lagrangian function $L(\mu)$ satisfies the integrality property if it has an integer optimal solution for every choice of μ even if we relax the integrality restriction on the variable X.*

Fisher (1981) has proved that for problems satisfying the integrality property, solving the Lagrangian multiplier problem (L) is equivalent to solve the linear relaxation of the problem (P). In these situations, though Lagrangian relaxation technique provides no better a bound than the linear programming relaxation, yet, the Lagrangian relaxation technique might be of considerable value, because solving the Lagrangian multiplier problem (L) might be more efficient than solving the linear programming relaxation directly. It will be an efficient solution strategy where Lagrangian subproblems are network flow problems in nature. The Lagrangian relaxation can exploit the core network substructures - shortest path, minimum cost flow, spanning tree, assignment and other problems. Usually, we choose the constraints to relax so that we can exploit underlying structure of the relaxed problem.

Our work is concentrated to identify all such applications in a variety of areas where Lagrangian relaxation of complicating constraints results in network flow subproblem (or subproblems). These Lagrangian

subproblems can be solved efficiently by network flow algorithms for a fixed value of Lagrangian multiplier vector and we can get bounds for the optimal solution of the original problem. Ahuja, Magnanti and Orlin (1989) presented a literature survey of network flow problems and described some of the fastest algorithm for solving these network flow problems.

CHAPTER 3

APPLICATIONS OF SHORTEST PATH PROBLEMS

In this chapter, we report applications in which the relaxed Lagrangian subproblems are shortest path problems. The following is the list of applications reported:

1. Optimal routing of hazardous material vehicle
   (Ramgopalan, Batta and Karwan [1990])
2. Routing of a by-product pipe line
3. Optimal election campaign tour
   (Ramgopalan, Batta and Karwan [1990])
4. Optimal routing of a irrigation canal
5. Routing of a police car trip
   (Ramgopalan, Batta and Karwan [1990])
6. Approximating piecewise linear function
   (Imai and Iri [1986])
7. Multi-period assignment of consultants
   (Aronson [1986])
8. Multi-period assignment of machine in fixed layout problems
9. Optimal assignment of sales force
10. Traffic assignment in communication satellites
    (Balas [1983])
11. A shortest path problem with side constraints
    (Aneja et al. [1983] and Minoux [1989])

12. School bus scheduling

(Desrosiers, Sauve and Soumis [1988])

Application 3.1 : Optimal routing of hazardous material vehicle

Reference : Ramgopalan, Batta and Karwan (1990)

A city municipal corporation has to plan the route of a hazardous material vehicle to transport material from a waste collection centre to isolated disposal centre. As there is a risk of exposure to city population from hazardous material, the corporation wants to minimize the risk. Furthermore, city is divided into $\mathbb{K}$ mutually disjoint zones or areas labeled as $z_1,\ldots,z_{\mathbb{K}}$. The objective is to minimize the total risk of exposure to the population of the city with an equitable distribution of risk among all the zones of the city.

We define a city transportation network **G = (N,A)** with N as a set of nodes representing the junction of two roads, and A as an arc set representing roads of the city. Origin (O) is the waste collection centre and destination (D) is the waste disposal site. Between any two nodes i and j, arc (i,j) exists if a road connects the junction i and junction j. Furthermore, a road may pass through several zones $(z_1,\ldots,z_{\mathbb{K}})$ of the city. For each link $(i,j) \in A$ and $k \in \{1,\ldots,\mathbb{K}\}$, $C_{ij} \ge 0$ represents the global risk associated with travel on arc (i,j), and $\pi_{z_k}(i,j) \ge 0$ represents the risk to zone z_k associated with travel on arc (i,j) such that :

$$\sum_{k=1}^{\mathbb{K}} \pi_{z_k}(i,j) = C_{ij} .$$

The corporation wants to have an equitable distribution of the risk among the various zones, that is, risk between any two arbitrary zones should be under a threshold of μ (predetermined) for the minimum risk path of the hazardous material vehicle.

The problem can be formulated as an equity constrained shortest path problem as follows:

$$\text{minimize} \sum_i \sum_j C_{ij} X_{ij}$$

subject to:

$$\sum_i \sum_j (\pi_{z_a}(i,j) - \pi_{z_b}(i,j)) X_{ij} \le \mu \quad \text{for all } a,b \in 1,\ldots,K, \tag{3.1a}$$

$$\sum_{\{j:(i,j)\in A\}} X_{ij} - \sum_{\{j:(j,i)\in A\}} X_{ji} = \begin{cases} 1 & \text{if } i = O \\ 0 & \text{if } i \ne O,\ i \ne D, \\ -1 & \text{if } i = D, \end{cases} \tag{3.1b}$$

$$X_{ij} = \begin{cases} 1 & \text{if arc } (i,j) \text{ is in optimal solution} \\ 0 & \text{otherwise.} \end{cases} \tag{3.1c}$$

Constraint set (3.1a) ensures equitable distribution of risk between any two arbitrary zones. For given $\mathbb{K}$ zones, there will be $(\mathbb{K}^2-\mathbb{K})$ constraints of type (3.1a). Constraint set (3.1b) and (3.1c) are shortest path constraints.

The problem is a constrained shortest path problem because of the presence of additional constraint set (3.1a). The model can be solved as a shortest path problem, if we relax constraint set (3.1a) by assigning Lagrange multipliers $U_{z_a z_b}$ (≥ 0) to each constraint in this set and forming a Lagrangian subproblem as follows:

$$\min L(x,u) = \sum_i \sum_j C_{ij} X_{ij} + \sum_{z_a} \sum_{z_b} U_{z_a z_b} \left(\sum_i \sum_j (\pi_{z_a}(i,j) - \pi_{z_b}(i,j)) X_{ij} - \mu\right)$$

subject to the constraint sets (3.1b) and (3.1c).

Alternatively, we can say:

$$\min L(x,u) = \sum_i \sum_j \left[C_{ij} + \sum_{z_a} \sum_{z_b} U_{z_a z_b} (\pi_{z_a}(i,j) - \pi_{z_b}(i,j))\right] X_{ij} - \mu \sum_{z_a z_b} \sum U_{z_a z_b}$$

subject to the constraint sets (3.1b) and (3.1c).

For a given vector of multipliers u, $\mu(\sum_{z_a} \sum_{z_b} U_{z_a z_b})$ will be a constant. Hence, Lagrangian subproblem can be solved easily by finding a shortest path with modified link risks as :

$$C'_{ij} = (C_{ij} + \sum_{z_a} \sum_{z_b} U_{z_a z_b} (\pi_{z_a}(i,j) - \pi_{z_b}(i,j))).$$

such that :

$$C'_{ij} \geq 0 \text{ for feasible solutions.}$$

Application 3.2 : Routing of by-product pipeline

In an Industrial city, a chemical company is engaged in manufacturing of products whose production results in certain by-product gases at different stages of manufacturing cycle. The company wants to utilize these by-products, by transporting these gases to some other plants in the city. As these by-products gases can be used as utilities at other plants, the company may get some revenue. The company has to get permission from state pollution clearance board to lay underground pipeline network to connect to the demand plant.

The company wants to design a underground pipeline network in a way, to minimize the total risk of exposure of the gases to the city population, as well as an equitable distribution of the risk among all the zones/areas of the city. The problem can be formulated as an equity constrained shortest path problem like Application 3.1.

Sometimes, a total investment budget constraint for the construction of such a pipeline can also be added. Let a_{ij} be the expected investment to construct arc (i,j) and d be the total budget limit for construction. Then :

$$\sum_{i}\sum_{j} a_{ij}\, X_{ij} \le d. \tag{3.1d}$$

The problem can be solved as a shortest path problem by relaxing total investment constraint (3.1d) and equitable distribution constraint (3.1a), using Lagrangian multipliers $\lambda \ge 0$ and $U_{z_a z_b} \ge 0$, with modified arc costs as :

$$C'_{ij} = (C_{ij} + \sum_{z_a} \sum_{z_b} U_{z_a z_b} (\pi_{z_a}(i,j) - \pi_{z_b}(i,j)) + \lambda a_{ij}) .$$

where $C'_{ij} \ge 0$ for feasible solutions.

This application can be found easily in the case of process and chemical industries.

Application 3.3 : Optimal election campaign tour

Reference : Ramgopalan, Batta and Karwan (1990)

A political party has to plan the tour of its main leader before the mid-term elections. The party wants to maximize his voting potential throughout the country. Furthermore, it also wants to maintain an equity in the time, that he spends among all the geographical/political zones of the country. The country has been divided in several geographical/ political zones.

The optimal election campaign tour problem is having a similar structure that of Application 3.1. Each arc length C_{ij} represents the gained voting potential while traversing the arc (i,j) of the transportation network and $\pi_{z_a}(i,j)$ is the gained voting potential in zone z_a, if arc (i,j) passes through the zone z_a, such that :

$$\sum_{a=1}^{\mathbb{K}} \pi_{z_a}(i,j) = C_{ij}.$$

The objective function will be a maximization in nature, hence the resulting problem will be a longest path.

Application 3.4 : Routing of a irrigation canal

A state irrigation board of a developing country has to plan the route of a proposed irrigation canal to facilitate the irrigation in underdeveloped region of the state. The board wants to maximize coverage of the agricultural towns while maintaining equitable coverage of the districts in that region. The problem can be formulated as Application 3.1, with the only difference that the objective function will be maximization in nature. To avoid looping paths (cycles) in the problem, we can insist that on $X_{ij} = 0$ or 1, and $\sum_j X_{ij} \le 1$.

Some times, a budget constraint (3.1d) can also be added on the total investment limit for the construction of the canal.

Application 3.5 : Optimal routing of a police car trip

Reference : Ramgopalan, Batta and Karwan (1990)

A city police department has to plan the route of a police patrolling car trip, to maximize the coverage received by the citizens while equitably distributing its time among the communities for which it is responsible. The problem is also similar to Application 3.1.

Application 3.6 : Approximating piecewise linear function

Reference : Imai and Iri (1986)

Numerous applications encountered within many different scientific fields use piecewise linear functions. These functions contain a large number of breakpoints, hence they are expensive to store and manipulate. In these situations, it might be advantageous to replace the piecewise linear function by another approximating function that uses fewer break

points. By approximating the function, we will generally be able to save the storage space but we will, however, incur a cost because of the accuracy of the approximating function. Assume that, we want to approximate the function using a given number p of points. The objective of making approximation is, that we would like to make the best possible tradeoff between these conflicting costs and benefits such that the new function passes through a given number of points.

Let $f_1(X)$ be a piecewise linear function of a scalar X. We represent the function in the two dimensional plane; it passes through n points $a_1 = (X_1, Y_1)$, $a_2 = (X_2, Y_2), \ldots, a_n = (X_n, Y_n)$. Suppose that we have ordered the points so that $X_1 \le X_2 \le \ldots \le X_n$. We assume that the function varies linearly between every two consecutive points X_i and X_{i+1}. We consider the situation in which n is very large and for practical reasons we wish to approximate the function $f_1(X)$ by another function $f_2(X)$ that passes through not more than p points {inclduing a_1 and a_n}, such that $p < n$.

The approximated function results in the savings in storage space. Let α be the per unit cost associated with any single interval used for approximation. Furthermore, the error of an approximation is proportional to the sum of the squared errors between the actual data points and the estimated data points, i.e., the penalty is $\beta \left[\sum_{k=i}^{j} (f_1(X_k) - f_2(X_k))^2\right]$ for some constant β. The problem can be formulated as a constrained shortest path problem on a network $G = (N,A)$ with n nodes, numbered 1 through n. The arc (i,j) signifies that we approximate the linear segment of the function $f_1(X)$ between the points a_i, $a_{i+1}, \ldots, a_j$ by one linear segment joining the points a_i and a_j. The cost of the arc (i,j) will be given by :

$$C_{ij} = \alpha + \beta \left[\sum_{k=i}^{j} (f_1(X_k) - f_2(X_k))^2\right].$$

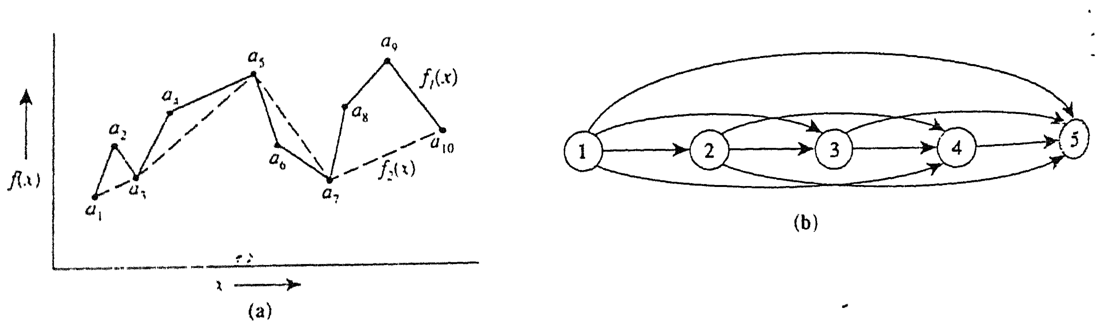

Fig 2 : Illustrating Application 3.6 : (a) approximating the function $f_1(X)$ passing through 10 points by the function $f_2(X)$; (b) corresponding shortest path problem.

Let X_{ij} be a zero-one variable, indicating whether arc (i,j) is included or not in the optimal solution. The problem can be formulated as :

$$\text{minimize} \sum_{(i,j)\in A} C_{ij}\, X_{ij}$$

subject to:

$$\sum_{\{j:(i,j)\in A\}} X_{ij} - \sum_{\{j:(j,i)\in A\}} X_{ji} = \begin{bmatrix} +1 & \text{for } i = 1, \\ 0 & \text{for } i \neq 1,n, \\ -1 & \text{for } i = n, \end{bmatrix} \tag{3.6a}$$

$$\sum_{(i,j)\in A} X_{ij} \leq p-1 \text{ for all } i,j \in N, \tag{3.6b}$$

$$X_{ij} = \begin{cases} 1, & \text{if arc } (i,j) \text{ is included,} \\ 0, & \text{otherwise.} \end{cases} \tag{3.6c}$$

Constraint (3.6b) ensures that the function should be approximated in a manner, such that there should not be more than p points in the new function. This constraint can be relaxed, using a Lagrangian multiplier $\lambda \geq 0$. The relaxed problem will be

$$\text{minimize } L(x,\lambda) = \sum_{(i,j)\in A} (C_{ij} + \lambda)\, X_{ij} - \lambda(p-1).$$

subject to the constraint set (3.6a) and (3.6c).

For a fixed value of λ, $\lambda(p-1)$ will be a constant. The relaxed problem can be solved as a shortest path problem having modified arc costs as $C'_{ij} = C_{ij} + \lambda$. Note that here, we assume that the new function must pass through the points 1 and n.

Application 3.7 : Multiperiod assignment of consultants

Reference : Aronson (1986)

A consultancy firm provides project monitoring and project assistance services to its clients. The firm appoints its consultants at different client offices depending upon the project requirements. The firm wants to

determine the optimal assignment of its consultants over a fixed time period T. With each assignment of consultant i to client j in any time period t ($t = 1,\dots,T$), C_{ijt} is the assignment cost. The assignment cost need not to be equal in each time period because of varying skills of consultants and different requirements of projects. Furthermore, let S_{ijkt} be the transferring cost (training, moving, resettling) of consultant i from client j in period t to client k in period t+1.

The optimal multiperiod assignment of consultants problem can be described as an integer multicommodity flow problem. We define a network G = (N,A), consisting of a set of nodes N and a set A of ordered pair of nodes (p,s), called arcs. Each consultant is represented as a commodity. We also assume that the number of consultants and number of clients are both equal to n.

For the network G = (N,A), let $N_p^+ \subseteq N$ denote the set of nodes $s \in N$ for which arc set $(p,s) \in A$, i.e., the set of nodes which have arcs pointing away from node p; and $N_p^- \subseteq N$ denote the set of nodes $s \in N$ for which the arc $(s,p) \in A$, i.e, the set of nodes which have arcs pointing toward node p. Furthermore, for each commodity i, $i = 1,\dots,n$, with each arc $(p,s) \in A$ is associated a cost per unit flow C_{ps}^i which will be set to either C_{ijt} or S_{ijkt}. We define the node set to be N = $\{0,1,\dots,(2T-2)n+1\}$ and arc set to be the union of 2T-1 disjoint sets; T sets of assignment arcs (A_t) and T-1 sets of transfer arcs (T_t). The node requirements for $i = 1,\dots,n$ are $r_0^i = +1$, $r_{(2T-2)n+1}^i = 1$, and $r_p^i = 0$ for $p = 1,\dots,(2T-2)n$. The problem can be formulated as follows:

$$\text{minimize} \sum_{i=1}^{n} \sum_{(p,s)\in A} C_{ps}^i \, X_{ps}^i$$

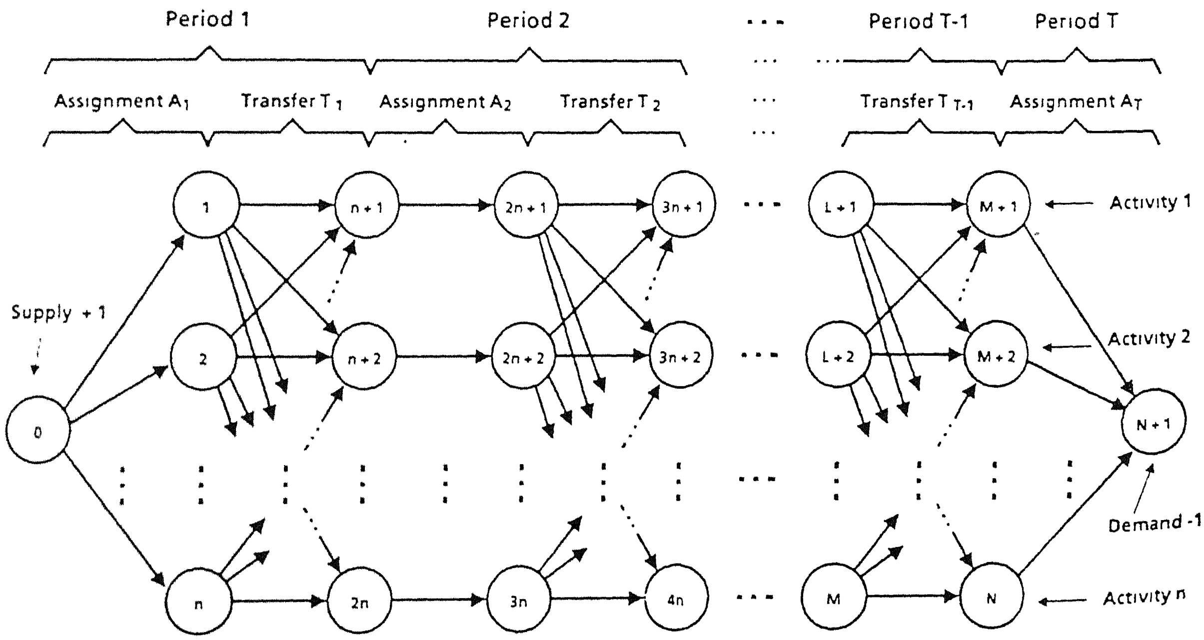

Fig 1. Graphical representation of a single commodity of the multicommodity network for the multiperiod assignment problem (Application 3.7).

{ Above $L = (2T-4)n$, $M = (2T-3)n$, and $N = (2T-2)n$ }

subject to

$$\sum_{s\in N_p^+} X_{ps}^i - \sum_{s\in N_p^-} X_{sp}^i = r_p^i, \quad p \in N,\ i = 1,\dots,n, \qquad (3.7a)$$

$$\sum_{i=1}^{n} X_{ps}^i \le 1, \ (p,s) \in A_t \ , \ t = 1,\dots,T, \qquad (3.7b)$$

$$X_{ps}^i \ge 0, \ \text{ and integer}, \ (p,s) \in A, \ i = 1,\dots,n. \qquad (3.7c)$$

The mutual capacity constraints (3.7b) prohibits the assignment of a client to more than one consultant in each period. We propose to relax the mutual capacity constraints (3.7b) by using nT Lagrangian multipliers ($\lambda_{ps} \ge 0$). The relaxed problem will be to

$$\text{minimize} \ \sum_{i=1}^{n} \sum_{(p,s)\in A} C_{ps}^i X_{ps}^i + \sum_{(p,s)\in A_t} \lambda_{ps}\left(\sum_{i=1}^{n} X_{ps}^i - 1\right)$$

subject to (3.7a) and (3.7c).

or, equivalently,

$$\min \sum_{i=1}^{n} \sum_{(p,s)\in A_t} (C_{ps}^i + \lambda_{ps}) X_{ps}^i + \sum_{i=1}^{n} \sum_{(p,s)\in T_t} C_{ps}^i X_{ps}^i - \sum_{(p,s)\in A_t} \lambda_{ps}$$

subject to (3.7a) and (3.7c).

The resulting objective function for the Lagrangian subproblem has a cost of $C_{ps}^i + \lambda_{ps}$ for each assignment arc and C_{ps}^i for each transfer arc. Since none of the constraints in this problem contains the flow variables (X_{ps}^i) for more than one of the consultants, the problem can be decomposed into n pure independent shortest path subproblems.

Application 3.8 : Multi-period assignment of machines in fixed-layout problems

A manufacturing division has n machines, having different efficiencies or operating costs (C_{ijt} of Application 3.7) for different sets of task

requirements at n different locations over a time period of T. The tasks at various locations require similar type of work but with varying efficiency requirements and other work parameters. Furthermore, any machine can be operated at any location to perform the required task but with varying efficiency and costs. In addition, the cost of moving or transferring (S_{ijkt} of Application 3.7) these heavy machines from one task location to another location may be substantial. The objective of the problem is to find out a minimum cost assignment of the machines at different tasks locations over a time period T. The problem of optimal multi-period assignment of machines in fixed layout problems is also similar to Application 3.7. This problem can also be found easily at construction sites where mobile machinery is used at different locations and at docks where ship construction work is going on.

Application 3.9 : Optimal assignment of sales force

A marketing division of an organization wants to determine the minimum cost assignment of sales-managers to look after sales and customer services at different marketing territories over a time period of T. Based on good demand forecasting methods, they know job requirements and work load in these territories during each time period in advance. Furthermore, sales manager have varying marketing skills and different assignment costs. The optimal assignment of sales force problem aims at determining sales managers assignment to minimize the total assignment and transfer costs over a fixed period of time.

The problem is having a structure similar to that of Application 3.7.

Application 3.10 : Traffic assignment in communication satellites

Reference : Balas (1983)

Everyday a rapidly increasing volume of long distance TV, radio and

telephone communications is transmitted digitally via satellites. A high capacity communication satellite interconnects simultaneously scores of transmitting and receiving stations. One of the advanced techniques for operating such a satellite communication network is SS/TDMA (satellite-switched time division multiple access system) based on the use of highly directed spot beam antennas. Under this scheme, each transponder on board of satellite is allocated to a pair of ground stations for certain amount of time, based on the communication traffic between these two stations. Each transponder has different capacity. Such a set of allocations involving all the transponders is called a 'switch mode'. These allocations that make up a switch mode are changed simultaneously by an on-board switching facility. The whole sequence of switch modes that make up the schedule of allocation for a given period of time is called a frame.

The problem of optimal traffic assignment of communication satellite deals with the efficient scheduling of time slot allocations for a frame. As each transponder is having different capacity, the problem is to find a multi-period assignment of all transponders to various pairs of ground stations, based on the communication traffic.

The problem can be formulated similar to Application 3.7.

Application 3.11 : A shortest path problem with side constraints

In many real life problems we have to identify a shortest path from one node to another node with an additional constraint. This problem can be viewed as multiple criteria shortest path problem. Below, is the list of such applications :

(A) A minimum travel cost path between two cities subject to constraints on traversal time, energy consumption, **[Aneja et al. (1983)]**.

(B) Shortest path with fading out constraints for routing problems in telephone networks **[Minoux (1989)]**.

(C) Path of average minimal length with constraints on the variance in probabilistic graphs **[Minoux (1989)]**.

These problems can be formulated as shortest path problems having a side constraint on the other criteria, as follows :

$$\text{minimize} \sum_{(i,j)\in A} C_{ij} X_{ij}$$

subject to

$$\sum_{j\in N} X_{ij} - \sum_{j\in N} X_{ji} = \begin{cases} 1 \text{ if } i = \text{source}, \\ 0 \text{ if } i \neq \text{source}, \ i \neq \text{sink}, \\ -1 \text{ if } i = \text{sink} \end{cases} \tag{3.11a}$$

$$\sum_{(i,j)\in A} a_{ij} X_{ij} \leq b, \tag{3.11b}$$

$$X_{ij} \in (0,1). \tag{3.11c}$$

The constraint (3.11b) is a side constraint in the shortest path problem. The problem can be solved as a shortest path problem, by relaxing complicating constraint (3.11b), using a Lagrangian multiplier $\lambda > 0$. The Lagrangian subproblem will be as follows :

$$\text{minimize} \sum_{(i,j)\in A} (C_{ij} + \lambda\, a_{ij})\, X_{ij} - \lambda b$$

subject to the constraints (3.11a), (3.11c). For a fixed value of the multiplier $\lambda \geq 0$, λb will be a constant. The relaxed problem can be solved as a shortest path problem.

Application 3.12 : School bus scheduling problem

Reference : Desrosiers, Sauve and Soumis (1988)

School buses have to be scheduled over a set of trips. In the morning, a bus picks up students along its trip and delivers them to the school. In the afternoon, a trip begins at the school and ends at the stop where the last student is dropped off. Furthermore, it is also desirable to have a single bus sequentially cover several trips. Each locality where the students reside and every school has to be visited within a specified time interval. The problem of school bus scheduling aims at finding the minimum number of buses required to visit a set of nodes subject to the time window constraint, for a homogeneous fleet of buses located at a common depot. This problem was formulated as a network flow problem with additional time window constraints by Dersosiers et al. (1988).

Two Lagrangian relaxations of the model were studied by the Dersosiers (1988). In the first one, the time constraints are relaxed producing shortest path subproblems which are easy to solve, but the bound obtained is weak. In the second relaxation, constraints requiring each node to be visited are relaxed producing shortest path subproblems with time window constraints and integrality conditions.

CHAPTER 4

APPLICATIONS OF MINIMUM SPANNING TREE PROBLEMS

In this chapter we report applications in which the relaxed Lagrangian subproblems are either a undirected minimum spanning tree problem or a directed minimum spanning tree problem. We describe the following applications in this chapter:

1. Optimal drilling of oil wells
   (Akinc and Srikanth [1992])
2. Optimal tour of a health care delivery team
   (Akinc and Srikanth [1992])
3. Optimal tour of a maintenance team
   (Akinc and Srikanth [1992])
4. Tour of a mobile departmental store
5. Travelling buyer's problem
   (Akinc and Srikanth [1992])
6. Servicing of a wide forest area
   (Akinc and Srikanth [1992])
7. Design of centralized computer network
   (Volgenant [1989] and Gavish [1982])
8. Multi-drop terminal layout in teleprocessing design
   (Boorstyn and Frank [1985])
9. Reconstructing of a utility network after natural disaster
   (Shogan [1983])

10. Optimal offshore pipeline network design

(Rathforb, Frank, Rosenbaum and Steglits [1970])

Application 4.1 : Optimal drilling of oil wells

Reference : Akinc and Srikanth (1992)

Oil and gas commission of a country wants to drill exploratory oil wells in a newly discovered oil field to determine the size and other properties of oil field. To drill such exploratory wells, platforms are needed at these locations for mounting of oil rigs. Furthermore, it is usually possible to drill several such wells from a platform that is stationary at a point. As the construction cost of these platforms are quite substantial; therefore, significant economies may be realized by using towable platorms rather than stationary ones and drilling of oil-wells from these platforms. The drilling cost will be related to the distance and the angle of well to be drilled from the platform. Thus a towable platform will act like a mobile service unit that will visit some oil well locations physically, but not all the locations, and we will drill the rest of the locations from afar (at a cost) while stationed at a nearby locations. The OGC wants to determine the subset of all n oil well locations the towable platform should actually visit, the sequence of these visits and subset of all the points the platform should drill from each location on its path so that the total cost of drilling is minimized.

Let M_{ij} denote the cost of moving of a towable platform from location i to location j (sum of towing, set-up and take down cost of the platform) and P_{ij} denote the cost of drilling at location j while towable platform is at location i. Furthermore, m_i denotes the set of oil well locations the platform can move to from location i and p_i denotes the set of oil well locations that can be drilled from location i. S is the set of all the n

locations of oil wells to be drilled.

Let us define a zero-one variable X_{ij} indicating whether or not the platform moves from location i to location j, and Y_{ij} indicating whether a location j is serviced by the platform stationed at location i or not. We add one dummy location 0 in the set S, such that

$$S' \equiv S \cup 0.$$

Furthermore, $M_{0i} = M_{i0} = 0$,for all $i \in S$ and $P_{0i} = P_{i0} = \infty$, for all $i \in S$.

The problem can now be formulated as :

$$\text{minimize} \sum_{i} \sum_{j \in S'} M_{ij} X_{ij} + \sum_{i} \sum_{j \in S'} P_{ij} Y_{ij}$$

subject to

$$\sum_{\substack{j \in S \\ i \neq j}} Y_{ij} - |p_i| \sum_{\substack{j \in S' \\ i \neq j}} X_{ij} \leq 0 \qquad \text{for all } i \in S, \qquad (4.1a)$$

$$\sum_{i \in S'} (Y_{ij} + X_{ij}) = 1 \qquad \text{for all } j \in S, \qquad (4.1b)$$

$$\sum_{i \in S'} X_{ji} - \sum_{j \in S'} X_{ij} = 0 \qquad \text{for all } i \in S, \qquad (4.1c)$$

$$\sum_{j \in S} X_{0j} = \sum_{j \in S} X_{j0} = 1, \qquad (4.1d)$$

$$\sum_{j \in S_k} \sum_{i \in S_k} (X_{ij} + Y_{ij} \leq |S_k| - 1 \text{ for } S_k \subseteq S, \qquad (4.1e)$$

$$X_{ij}, Y_{ij} = 0, 1, \qquad \text{for } (i,j) \in S', \qquad (4.1f)$$

Constraint set (4.1a) prevents any drilling from a location unless the towable platform visits it. Constraint set (4.1b) requires that each location either be visited by the towable platform or else drilled from another location. Constraint sets (4.1c), (4.1d) and (4.1e) together ensure that the path of the visits by towable platform starts and finishes at dummy location 0, is contiguous, and is free of any subtours.

The problem can be solved by relaxing the constraint set (4.1a) and (4.1c), using two Lagrangian multipliers $\lambda_i \leq 0$ and μ_i unrestricted in sign, as follows :

$$L(\lambda,\mu) = \min \sum_i \sum_{j\in S'} M_{ij} X_{ij} + \sum_i \sum_{j\in S'} P_{ij} Y_{ij} + \sum_{i\in S} \lambda_i \Big(|p_i| \sum_{\substack{j\in S' \\ j\neq i}} X_{ji} - \sum_{\substack{j\in S \\ j\neq i}} Y_{ij} \Big)$$

$$+ \sum_{i\in S} \mu_i \left(\sum_{j\in S'} X_{ij} - \sum_{j\in S'} X_{ji} \right)$$

subject to the constraints (4.1b), (4.1d), (4.1e), and (4.1f). Alternatively, we can say :

$$\min \sum_i \sum_{j\in S'} \left(M_{ij} + \lambda_j \, |p_j| + \mu_i - \mu_j \right) X_{ij} + \sum_i \sum_{j\in S} (P_{ij} - \lambda_i) \, Y_{ij}$$

subject to the constraints (4.1b), (4.1d), (4.1e), and (4.1f).

We can also write the objective function as follows

$$L(\lambda,\mu) = \min \sum_i \sum_{j\in S'} \bar{M}_{ij} X_{ij} + \sum_i \sum_{j\in S} \bar{P}_{ij} Y_{ij}$$

where $\bar{M}_{ij} = (M_{ij} + \lambda_j \, |p_j| + \mu_i - \mu_j)$ and

$\bar{P}_{ij} = (P_{ij} - \lambda_i)$.

The optimal solution of this problem has the following properties :

(a) for any (i,j), X_{ij} and Y_{ij} can not be equal to one simultaneously due to the constraint set (4.1b);

(b) $\bar{P}_{ij} > \bar{M}_{ij} \implies Y_{ij} = 0$; and

(c) $\bar{M}_{ij} > \bar{P}_{ij} \implies X_{ij} = 0$.

The Lagrangian subproblem can therefore be simplified further by defining a new set of variable Z_{ij}, where

$$Z_{ij} = X_{ij} + Y_{ij}$$

with associated objective function coefficients as :

$$C_{ij} = \min \{\bar{P}_{ij}, \bar{M}_{ij}\}.$$

For a given set of λ_i and μ_i, the simplified Lagrangian subproblem will be:

$$L(\pi,\mu) = \min \sum_{i} \sum_{\substack{j \in S' \\ i \neq j}} C_{ij} Z_{ij}$$

subject to

$$\sum_{\substack{i \in S' \\ i \neq j}} Z_{ij} = 1 \qquad \text{for } j \in S \tag{4.1g}$$

$$\sum_{j \in S'} Z_{0j} = \sum_{j \in S'} Z_{j0} = 1 \tag{4.1h}$$

$$\sum_{\substack{i,j \in S_k \\ i \neq j}} Z_{ij} \leq |S_k| - 1 \qquad \text{for } S_k \subseteq S \tag{4.1i}$$

$$Z_{ij} = 0 \text{ or } 1 \qquad \text{for all } i,j. \tag{4.1j}$$

The optimal solution to the Lagrangian subproblem **(L)** will be a set of n+1 arcs consisting of the n arcs in the directed minimal spanning tree on the nodes in S, with the root node 0 restricted to degree 1, plus a single arc returning to node 0. The problem can be solved in polynomial time. (Tarjan [1977]; Lawler [1976]).

Application 4.2 : Optimal tour of a health care delivery team

Reference : Akinc and Srikanth (1992)

Health department of a underdeveloped country has to plan the route of a mobile health care delivery team in a particular region of the state. The team will make a tour of subset of all the villages in the region and those needing care at other villages may come for service while the team is temporarily stationed at some close-by village. The problem can be modeled as in Application 4.1.

Application 4.3 : Optimal tour of a maintenance team

Reference : Akinc and Srikanth (1992)

A maintenance team has to render routine service of maintaining a communication system at geographically dispersed points. As the total cost of visiting each point increases, significant economies may be obtained by organizing a small maintenance unit (or units) and co-ordinating some of their resource needs at temporary and movable bases stationed at some nearby locations. These bases may also schedule the activities of small units, house hard-to-move equipments, maintain communication with headquarters and provide lodging.

The maintenance team wants to determine the subset of all location that it should physically visit, the sequence in which these visits should be made and subset of all locations the maintenance team should service from each location on its path so that the total cost is minimized.

The application is having structure similar to that in Application 4.1.

Application 4.4 : Tour of a mobile departmental store

A consumer co-operative society has to plan the route of a mobile departmental store through various parts of the city. Because of the time constraint, the store can't visit all the localities physically rather it will visit a subset of all localities of the city and customers needing some consumer goods at other localities may come for purchasing while store is stationed temporarily at some close-by locality. The problem is also similar to Application 4.1. The cost of providing customer services (P_{ij} of Application 4.1) indirectly from some other nearby locality can be measured as expected sales loss because of not serving these customers directly.

Application 4.5 : Travelling buyer's problem

Reference : Akinc and Srikanth (1992)

A travelling buyer has to purchase a number of items from various vendors located at different geographical points. These vendors stock a subset of all the items and sell them at varying prices. The problem is to determine the subset of vendors and the sequence of the visits to these by the buyer so that the sum of travel and purchase cost is minimized. The traveling buyer's tour problem can be modeled as Application 4.1. The tour of the buyer may be modeled as the moves of a mobile service unit (buyer), while the items to be bought while at some vendor's location may be considered as services to be performed.

Application 4.6 : Servicing of a wide forest area

Reference : Akinc and Srikanth (1992)

A forest department has to plan the optimal servicing plan of a wide forest area. The services are provided as a routine service by a team of forestry personnel to the remote parts of the forest. In rendering these services, economies may be obtained by a "satellite" service unit (or units) and coordinating their needs for equipment etc. from temporary and movable bases. The problem is similar to Applications 4.1 and 4.2.

Application 4.7 : Design of centralized computer network

Reference : Volgenant (1989) amd Gavish (1982)

In a centralized computer network, several terminals at various sites have to be connected to a central computer, with a minimum amount of wire. Furthermore, the size of the terminals is such that the number of wires incident to a terminal k can not exceed a number r_k. Let C_{ij} be the amount of wire required to connect two terminals i and j. Furthermore, we define a set of zero-one variables X_{ij}, which attains the value of one if arc

(i,j) is included and zero otherwise; and Y_{ij} is a set of variable that specifies flow on the arc connecting terminals i and j.

The problem of designing an optimal centralized computer network will be a degree constrained spanning tree with a root in the central site that spans over the other sites and with each terminal k restricted to a degree r_k and has a minimal connection wire length. Let $I = \{2,\ldots,n\}$ denote the set of different terminal locations and the central computer site is denoted as node 1, such that

$$\overline{I} = I \cup 1.$$

The problem can be formulated as :

$$\text{minimize } z = \sum_{i=2}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} C_{ij} X_{ij}$$

subject to

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} X_{ij} = 1 \qquad \text{for } i = 2,\ldots,n, \qquad (4.7a)$$

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} Y_{ij} - \sum_{\substack{j=2 \\ j\neq i}}^{n} Y_{ji} = 1 \qquad \text{for } i = 2,\ldots,n, \qquad (4.7b)$$

$$Y_{ij} \leq (n-1)\, X_{ij} \qquad \text{for } i = 2,\ldots,n,$$

$$j = 1,2,\ldots,n \text{ and } i \neq j, \qquad (4.7c)$$

$$Y_{ij} \geq 0 \text{ and } X_{ij} = 0 \text{ or } 1 \qquad \text{for all } i,j \in \overline{I}, \qquad (4.7d)$$

$$\sum_{i=2}^{n} X_{ik} + \sum_{i=1}^{n} X_{ki} \leq r_k \qquad \text{for all } k \in S. \qquad (4.7e)$$

Constraint sets (4.7a), (4.7b), (4.7c) and (4.7d) ensure the spanning tree structure of the network and constraint (4.7e) ensures that for any terminal k, its degree should not exceed $r_k \geq 1$. The subset S is a set of

all nodes on which degree constraint has been imposed.

The problem can be solved as a minimal spanning tree problem by multiplying the constraints in (4.7e) by a vector of Lagrangian multiplier $\{\lambda\}$ and adding them to the objective function. This gives the following Lagrangian subproblem :

$$L(\lambda) = \min \sum_{k\in S} \lambda_k r_k + \sum_{i=2}^{n} \sum_{j=1}^{n} \bar{C}_{ij} X_{ij}$$

subject to the constraints (4.7a), (4.7b), (4.7c) and (4.7d), where

$$\bar{C}_{ij} = \begin{cases} C_{ij} & \text{for } i \notin S \text{ and } j \notin S, \\ C_{ij} - \lambda_i & \text{for } i \in S \text{ and } j \notin S, \\ C_{ij} - \lambda_j & \text{for } i \notin S \text{ and } j \in S, \\ C_{ij} - \lambda_i - \lambda_j & \text{for } i \in S \text{ and } j \in S, \end{cases}$$

For a fixed vector of multiplier λ, the problem will be a minimal spanning tree problem.

Application 4.8 : Multi-drop terminal layout in teleprocessing design

Reference : Böorstyn and Frank (1985)

Computer communication networks are becoming a prevalent part in the corporate environment. A problem, which is common to many computer communication network design problem is to find a minimal cost spanning tree that connects a set of known terminal sites to the central site. In such networks, generally a polling scheme to regulate the traffic is used so that the terminals can send messages over common branches. The tree structure of a teleprocessing network can be divided into subtrees rooted at the central site. Each subtree is called a multidrop line. The terminals on each multidrop line are regulated by a polling regimen. Furthermore, each port of the communication controller in the central site

that deals with a multidrop line has a limited capacity on the amount of traffic that it can handle. So, the capacity of the link connecting each multi-dropline to the central site must be less than or equal to the limited capacity of each port. The problem was formulated as a capacitated minimum spanning tree problem by **Gavish [1985]**. A Lagrangian relaxation was used to solve the problem as a degree constrained minimum spanning tree problem.

Application 4.9 : Reconstructing of a utility network after natural disaster

Reference : Shogan (1983)

Cities, states, and countries rely heavily on a variety of utility networks : energy networks (such as electrical or natural-gas networks), communication networks (such as telephone, telegraph, or computer networks), transportation networks (such as highway or railrood networks), and water networks (such as networks for the distribution of portable water or the removal of sewage). Many important networks do not have a tree structure, and, after sustaining damage from some natural disaster (such as from earthquake), the long run goal for such a network will be to rebuild the network so that it is as good or better than it was prior to the earthquake. However, a less ambitious short run goal may simply be to allocate scarce resources (Labor, equipment, spare parts etc.) to the repair process in such a way as to restore some "minimal level of service" as quickly and economically as possible.

Let N be the set of nodes having indices 1,2,...,n; node 1 represent the source having an infinite supply of a commodity, and every other node p is a sink node having a known demand for that commodity. There are $m \equiv \frac{1}{2} n(n-1)$ arcs in the network before the natural disaster. Associated with each arc $j (1 \leq j \leq m)$, e_j is the flow capacity of arc j. The minimal level

of service after natural disaster can be provided by constructing a spanning tree (T) when only a single source exists, or, by a spanning forest (when multiple sources exist); so that every demand node would have access to a source. Furthermore, the restortion of arc $j(1 \le j \le m)$ to its former capacity e_j would cost c_j dollars and would consume a_{ij} $(1 \le i \le K)$ units of the limited quantity of b_i units of resource i. The problem can be formulated as a resource constrained spanning tree problem, as follows :

$$\min \sum_{j=1}^{m} C_j X_j .$$

subject to

$$\sum_{j=1}^{m} a_{ij} X_j \le b_i , \qquad 1 \le i \le K, \tag{4.9a}$$

$$X \in T. \tag{4.9b}$$

Here, X denotes a binary vector whose jth component X_j equals 1 if arc j is included in the minimal spanning tree and equals 0 otherwise. The problem can be solved as a minimal spanning tree problem by relaxing the constraint set (4.9a), and associating non-negative Lagrangian vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_K)$ with each constraint. The relaxed problem will be :

$$L(\lambda) = \min \sum_{j=1}^{m} C_j X_j + \sum_{i=1}^{K} \lambda_i \left(\sum_{j=1}^{m} a_{ij} X_j - b_i \right)$$

subject to (4.9b),

or, equivalently,

$$\text{minimize} \sum_{j=1}^{m} \left(C_j + \sum_{i=1}^{K} \lambda_i a_{ij} \right) X_j - \sum_{i=1}^{K} \lambda_i b_i$$

subject to (4.9b).

For a fixed value of multipliers λ_i, $\sum_{i=1}^{K} \lambda_i b_i$ will be a constant and the subproblem will be a minimum spanning tree.

Application 4.10 : Optimal design of offshore pipeline network

Reference : Rathforb, Frank, Rosenbaum and Steglits (1970)

Several offshore natual gas fields have to be connected to an onshore separation and compression plant. The objective is to find out a minimum cost spanning tree structure that will connect all the fields to the onshore plant. Sometimes, a constraint on the amount of pipe available can be imposed in such a design. The problem can be formulated as a resource constrained minimal spanning tree problem like Application 4.9. A detailed model having certain other constraints on several other criterias has been discussed by B. Rathforb et al. (1970). The relaxed subproblem can be solved as a spanning tree problem, as in Application 4.9.

CHAPTER 5

ADDITIONAL APPLICATIONS

In this chapter, we report applications whose Lagrangian relaxations result in more than one Lagrangian subproblems. In addition to these, we will also report the applications in which relaxed Lagrangian subproblems will be network flow problems other than shortest path and minimum spanning tree problems. These applications are as follows:

1. Optimal assignment of trainees
2. Core management of nuclear reactor

   (Gupta and Sharma [1981])
3. Job shop loading
4. Tour of a news paper vehicle
5. Optimal tour of a courier service personnel
6. Sequencing of a drilling job
7. Interconnection of local area networks

   (Fetterolf and Anandlingam [1991])
8. Electrical distribution system design

   (Current and Pirkul [1991])
9. Optimal design of mass transit system

   (Current and Pirkul [1991])
10. Optimal design of gas field pipeline network

    (Current and Pirkul [1991])
11. Transportation network design for resource extraction area

12. Transportation network design for a developing country
    (Current, Re-velle and Cohon [1986])
13. Optimal newspaper distribution network
    (Current, Re-velle and Cohon [1986])
14. Project scheduling with resource constraints
    (Christofides, Alvarez-valdes and Tamirit [1987])
15. Procurement of aviation fuels
    (Austin and Hogan [1976])
16. A forest industry problem
    (Jörnsten and Väbrand [1986])
17. A logistics planning system problem
    (Klingman, Mote and Philips [1986])

Application 5.1 : Optimal assignment of trainees

A heavy engineering company has to recruit workers as trainee-apprentice for one of its manufacturing division. The recruited workers have to be provided training for different tasks in that manufacturing division. The company has some yearly training budget to be spent on the training of new employees. Let C_{ij} be the expected revenue for the company and a_{ij} be the training expenditure, if ith trainee is assigned to jth job. Furthermore, the total number of trainees, n, are equal to the number of tasks to be assigned. The numbers C_{ij} and a_{ij} are known in advance because of the qualification and technical skills of the trainees. The optimal assignment of trainees problem can be formulated as a constrained assignment problem to find out the one-to-one assignment of trainees and jobs, with a view to maximize the expected revenue of the company. In case, the training budget should not exceed more than **d** rupees, this problem can be modeled as follows:

$$\text{maximize} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}$$

subject to

$$\sum_{i=1}^{n} X_{ij} = 1 \qquad \text{for } j = 1,\ldots,n, \tag{5.1a}$$

$$\sum_{j=1}^{n} X_{ij} = 1 \qquad \text{for } i = 1,\ldots,n, \tag{5.1b}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} X_{ij} \le d \tag{5.1c}$$

$$X_{ij} = \begin{cases} 1 \text{ , if ith worker is assigned on job j,} \\ 0 \text{ , otherwise.} \end{cases} \tag{5.1d}$$

Constraint set (5.1a) ensures the assignment of only one trainee for each task. Constraint set (5.1b) ensures assignment of only one task for each trainee. Constraint (5.1c) ensures that training expenditure should not exceed more than d rupees for the assignment.

The problem can be solved as an assignment problem, if we relax constraint (5.1c) by associating a Lagrangian multiplier $\lambda \ge 0$ with it. The relaxed problem will be as follows :

$$\max \sum_{i=1}^{n} \sum_{j=1}^{n} (C_{ij} + \lambda a_{ij}) X_{ij} - \lambda d.$$

subject to constraint set (5.1a), (5.1b) and (5.1d).

Application 5.2 : Core management of nuclear reactor

Reference : Gupta and Sharma (1981)

A nuclear power plant that uses pressurized water reactors (PWR) has to find out the optimal reactivity allocation of the given fuel assemblies having different burn up exposures. Let there be **n** fuel assemblies to be assigned to **n** possible locations in the reactor. The objective of the

problem is to find out the one to one assignment of fuel assemblies and reactor locations to maximize its reactivity, under the constraint imposed upon the peak load factor of power distribution. Let F_1 be the maximum core power distribution factor of the reactor. The problem can be formulated as follows :

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}.$$

subject to

$$\sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} X_{ij} \le F_1, \qquad (5.2a)$$

$$\sum_{i=1}^{n} X_{ij} = 1, \qquad \text{for } j = 1,\ldots,n, \qquad (5.2b)$$

$$\sum_{j=1}^{n} X_{ij} = 1, \qquad \text{for } i = 1,\ldots,n, \qquad (5.2c)$$

$$X_{ij} = \begin{cases} 1 & \text{if fuel assembly } i \text{ is assigned to reactor location } j, \\ 0 & \text{otherwise.} \end{cases} \qquad (5.2d)$$

Here, C_{ij} and d_{ij} are non-negative parameters. These can be calculated as follows:

E_i = exposure of fuel assembly i,

F = core power distribution form factor,

$(W_R)_i$ = Statistical weight of reactivity corresponding to location i,

$(W_F)_i$ = Statistical weight of form factor corresponding to location i,

$\bar{C}_{ij}$ = $(E_i - E_j)\,(W_R)_i$,

$\bar{d}_{ij}$ = $(E_i - E_j)\,(W_F)_i$,

C_{ij} = $\max\limits_{i,j} \bar{C}_{ij} - \bar{C}_{ij}$,

d_{ij} = $\bar{d}_{ij} - \min\limits_{i,j} \bar{d}_{ij}$,

F_1 = maximum power distribution form factor

= $F + n\ (\min d_{ij})$.

This problem is a constrained assignment problem, due to the constraint (5.2a). This can also be solved by using Lagrangian relaxation, as in Application 5.1.

Application 5.3 : Job shop loading

A job shop has a small group of similar machines having varying capacity etc. The functional requirement of various machines within a group is same, though their processing time, capacity, power consumption vary. Due to the nature of job shop, each such group of machines receives workpieces of different sizes and shapes in the lots. Let $\mathbf{C}_{ij}$ be the cost of assigning workpiece i to machine j and $\mathbf{t}_{ij}$ be the processing time for such assignment. The job shop loading problem aims to minimize the cost of assignment such that the total processing time of each lot should be within available 'd' machine hours.

The problem is similar to Application 5.1 except that the objective function will be minimization in nature.

Application 5.4 : Tour of newspaper vehicle

A newspaper printing plant has to determine the minimum distribution cost tour of its newspaper vehicle that will transport and deliver newspapers to certain cities of a particular zone, near to the printing plant.

The vehicle will start from its home base (printing plant), node 1, visit each of several cities in that zone, represented by nodes 2,...,n, exactly once and return home base after delivering newspapers, doing so at the lowest possible travel cost. Furthermore, to ensure good delivery, a time constraint on the total travel time of the vehicle is also imposed,

time constraint on the total travel time of the vehicle is also imposed, so that demand nodes can get newspapers as early as possible.

The problem can be formulated as travelling salesman problem with an additional constraint on the total travel time of newspaper vehicle. Let C_{ij} denote the cost of traveling and t_{ij} denote the travel time from city i to city j. Furthermore, plant wants that vehicle trip should not be more than T hours. Let us define a zero-one variable Y_{ij} indicating whether or not the vehicle travels from city i to city j, a flow variable X_{ij} on each arc (i,j) and assume that vehicle has n-1 units available at node 1 (home base) and it has to deliver 1 unit to each of the other nodes. The problem can be formulated as :

$$\text{minimize} \sum_{(i,j)\in A} C_{ij} X_{ij} .$$

subject to

$$\sum_{1\le j\le n} Y_{ij} = 1 \qquad \text{for all } i = 1,2,\dots,n, \tag{5.4a}$$

$$\sum_{1\le i\le n} Y_{ij} = 1 \qquad \text{for all } j = 1,2,\dots,n, \tag{5.4b}$$

$$\sum_{j} X_{ij} - \sum_{j} X_{ji} = \begin{cases} n-1 & \text{for } i = 1, \\ -1 & \text{for } i \ne 1, \end{cases} \tag{5.4c}$$

$$X_{ij} \le (n-1)\, Y_{ij} \qquad \text{for all } (i,j) \in A, \tag{5.4d}$$

$$\sum t_{ij} Y_{ij} \le T, \tag{5.4e}$$

$$X_{ij} \ge 0 \text{ and integer, for all } (i,j) \in A, \tag{5.4f}$$

$$Y_{ij} = \begin{cases} 1, & \text{if vehicle visits arc } (i,j), \\ 0, & \text{otherwise} \end{cases} \tag{5.4g}$$

The constraint (5.4a) and (5.4b) ensures that exactly one arc leaves and enters any node i, i.e., integer solution satisfying (5.4a) and (5.4b)

will be the union of disjoint cycles. Constraint (5.4c) is flow balance constraint and ensures connectivity of the network, constraint (5.4d) are redundant if $Y_{ij} = 1$ as no arc need ever carry more than (n-1) units of flow. Constraint (5.4e) ensures that the total trip time of vehicle should not exceed more than T hours.

The problem can be solved by attaching Lagrangian multipliers $\mu_{ij} \geq 0$ with the constraint (5.4d) and λ with constraint (5.4e) and bring them into objective function, giving the Lagrangian objective function :

$$\min \sum_{(i,j)\in A} [C_{ij} - (n-1)\ \mu_{ij} + \lambda t_{ij}]\ Y_{ij} + \sum_{(i,j)\in A} \mu_{ij}\ X_{ij} - \lambda T$$

subject to (5.4a), (5.4b), (5.4c), (5.4f) and (5.4g).

The relaxed problem decomposes into two subproblems :

(i) an assignment subproblem in variable Y_{ij}

$$\min \sum [C_{ij} - (n-1)\ \mu_{ij} + \lambda\ t_{ij}]\ Y_{ij}$$

subject to (5.4a), (5.4b) and (5.4g).

(ii) a minimum cost flow problem

$$\min \sum_{(i,j)\in A} \mu_{ij}\ X_{ij}$$

subject to (5.4c) and (5.4f).

For fixed value of λ, λT will be constant.

These subproblems are network flow problems.

Application 5.5 : Optimal tour of courier service personnel

A courier service personnel has to deliver various packets of letters at several cities, collected from the home city. The courier service wants to find out minimum cost travel tour, with a time constraint on the travelling done by employees. This problem will be similar to Application

4.1. Here, the cities, to be visited by messengers are connected through transportation network and are in the same zone so that the same person can travel through all the cities easily. Such problem may arise when different transportation modes are available between two cities. Furthermore, a quick service mode will be available at higher traveling cost to the courier service personnel.

Application 5.6 : Sequencing of a drilling job

A workpiece requires several holes of different diameters to be drilled on it. After drilling of each hole, we have to reset the machine (i.e. changing and positioning of the drill tool, clean the workpiece etc.), incurring a set up time. Let C_{ij} denote the set up time between drilling of two holes **i** and **j**. The objective is to find out the optimal processing sequence of the holes that will minimize the total set up time. The problem can be formulated as the traveling salesman problem - the "machine" which functions as the "salesman", needs to visit the jobs in the time effective manner. This application will be similar to Application 5.4 except constraint (5.4e) will not be present and the objective will be to minimize the total set up time.

Application 5.7 : Interconnection of local area networks
Reference : Fetterolf and Anandlingam (1991)

Local area networks (LANs) are popular means of interconnecting information systems in corporate environments. LANs are used to interconnect PCs, workstations, mini and mainframe computers, fileservers, database servers and printers to share the benefits of multi-user systems. In big organizations, multiple LANs (as many as 10 or 20) of varying types (Ethernets or token rings), capacities and costs are connected through high speed devices, called bridges, to monitor the transmission of datapackets

among the LANs.

The problem of optimal interconnection of LANs deals with the design of a LAN-LAN internetwork, with a given set of LANs, a traffic matrix defining host to host internetwork traffic, and a set of bridges of varying cost and capacities that will minimize the total cost of wiring and bridges costs. It will also satisfy following requirements :

a) Network flows should satisfy traffic matrix, i.e., requirement of each user to receive or send data must be satisfied.
b) Total average flow of data packets into each LAN should be below the recommended maximum for that LAN and it should also be below the maximum filter rate of each bridge connected to that LAN, i.e., maximum capacity of each bridge to determine whether packets should be transferred or not.
c) The total average flow of data packets into each bridge should be below the maximum transfer capacity of that bridge between two LANs.
d) To prevent data packets from getting caught in loops, internetwork of LANs should be a spanning tree.

Let L denote the set of LANs and K denote the set of bridges of different capacity and cost. LANs are modeled as nodes of networks, and bridges are modeled as a set of arcs [(i,j),(j,i)]. As the LANs are fixed, node capacities are fixed. However, arc capacity is dependent on the type of bridge used between two LANs. Let t_{ij} represents inter LAN traffic from LAN i to LAN j. Furthermore, a datapacket that is sent from LAN o to LAN d, is defined as a commodity od and X_{ij}^{od} represents the average flow of commodity od over the bridge connecting LAN i and LAN j, such that the total flow across the bridge (i,j) will be :

$$X_{ij} = \sum_{o\in L} \sum_{d\in L} X_{ij}^{od} .$$

Let C_{ij} be the wiring and installation cost associated with using a bridge to connect LANs i and j and C^k be the cost of purchasing a bridge of capacity k. We define a zero-one variable Y^k_{ij} to denote whether a bridge of capacity k is placed between LANs i and j or not.

The problem will be as follows :

$$\min \sum_{k\in K} \sum_{i\in L} \sum_{j>i} (C_{ij} + C^k)\, Y^k_{ij} \,.$$

subject to

$$\sum_{j\in L} X^{od}_{ji} - \sum_{j\in L} X^{od}_{ij} = \begin{cases} t_{od} & \text{if } i = d \text{ (destination)} \\ -t_{od} & \text{if } i = 0 \text{ (origin)} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{for all } i \in L, \text{ for all } o \in L, \text{ for all } d \in L, \qquad (5.7a)$$

$$X_{ij} + X_{ji} \le \sum_{k\in K} g^k\, Y^k_{ij} \text{ for all } i \in L,\ j > i, \qquad (5.7b)$$

$$\sum_{i\in L} (X_{ij} + t_{ji}) \le (\ell^k - h_j)\, I(Y^k_{jn}) + h_j \quad \text{for all } j \in L,$$

$$\text{for all } n \in L, \text{ for all } k \in K, \qquad (5.7c)$$

$$\sum_{k\in K} \sum_{i\in L} \sum_{j>i} Y^k_{ij} = |L| - 1, \qquad (5.7d)$$

$$X_{ij} = \sum_{i\in L} \sum_{j\in L} X^{od}_{ij} \qquad \text{for all } i \in L, \text{ for all } j \in L, \qquad (5.7e)$$

$$X^{od}_{ij} \ge 0 \quad \text{for all } i \in L, \text{ for all } j \in L, \text{ for all } o \in L, \text{ for all } d \in L, \qquad (5.7f)$$

$$Y^k_{ij} \in \{0,1\} \text{ for all } i \in L, \text{ for all } j \in L, \text{ for all } k \in K. \qquad (5.7g)$$

Constraint (5.7a) are network flow conservation constraints. Constraint (5.7b) restricts the total flow across each bridge, where g^k is the bridge

transfer rate. In constraint (5.7c), ℓ^k is the filter rate of bridge k and h_j specifies the capacity of LAN j. This constraint restricts total flow in each LAN. Constraint (5.7d) ensures the spanning tree structure of the inter connection network.

If we relax (5.7b) and (5.7c) by using two Lagrangian multipliers λ^1_{ij} (for constraint (5.7b)) and λ^2_{jnk} (for constraint (5.7c)), the relaxed problem can be decomposed in two subproblems :

Subproblem (1) :

$$\min \sum_{k \in K} \sum_{i \in L} \sum_{j > i} (C_{ij} + C^k) Y^k_{ij} - \sum_{i \in L} \sum_{j > i} \sum_{k \in K} \lambda^1_{ij} g^k Y^k_{ij}$$

$$- \sum_{j \in L} \sum_{n \in L} \sum_{k \in K} \lambda^2_{jnk} (\ell^k - h_j) I(Y^k_{jn})$$

subject to (5.7d) and (5.7g).

To use the minimum spanning tree algorithms, it is essential to determine the cost of connecting i and j relative to subproblem 1. Let A_{ij} denote this cost for any arbitrary pair of LAN i and LAN j. The spanning tree requirement dictates that only one bridge type in the set K may be used to span i and j. Therefore the minimum cost A_{ij} will be

$$A_{ij} = \min_{k \in K} \left\{ C_{ij} + C^k - \lambda^1_{ij} g^k - \lambda^2_{ijk} (l^k - h_i) - \lambda^2_{jik} (l^k - h_j) \right\}$$

This subproblem will be a spanning tree problem.

Subproblem (2) :

$$\min \sum_{i \in L} \sum_{j > i} \lambda^1_{ij} (X_{ij} + X_{ji}) + \sum_{j \in L} \sum_{n \in L} \sum_{k \in K} \sum_{i \in L} \lambda^2_{jnk} (X_{ij} + t_{ji}).$$

subject to (5.7a), (5.7e) and (5.7f).

This will be an uncapacitated multicommodity flow problem. This can also be viewed as minimum cost flow problem for each commodity od. In

order to minimize the cost of flows between O-D pairs on an uncapacitated network, it is required to route each O-D flow on the minimum cost path (shortest path) between origin and destination. Let W_{ij} denote the distance (cost) between each pair of nodes, such that

$$W_{ij} = \begin{cases} \lambda^1_{ij} + \sum_{n \in L} \sum_{k \in K} \lambda^2_{jnk} & \text{if } i < j, \\ \lambda^1_{ji} + \sum_{n \in L} \sum_{k \in K} \lambda^2_{jnk} & \text{if } j < i, \end{cases}$$

The shortest path between all pairs of nodes can be calculated by using all pair shortest path algorithms.

Application 5.8 : Electrical distribution system design

Reference : Current and Pirkul (1991)

A national electricity board has to determine the design of an electrical distribution system. They want to connect two major power generating plants with high voltage transmission lines to enhance the reliability of the system and to facilitate load sharing. Furthermore, to connect the end users of electricity with this transmission system, lower voltage distribution lines are required. Transformers and electrical substations at the interconnection of high voltage transmission line and lower voltage distribution lines are also needed. The problem of optimal electrical distribution system design identifies a two-level hierarchical network that includes a path of primary arcs (higher voltage line) between two power generating plants and all the demand nodes for electricity must either be at the site of a transshipment facility (transformer etc.), or to be connected to such a facility via a path of secondary arcs (lower voltage lines). All transshipment facilities must be located on the primary path.

We define an undirected network G = (N,A) with N as a set of n nodes, where node 1 and node n represent the power plants and other nodes

represent the demand nodes for the electricity, and A as a set of m arcs. Let C_{ij} and C'_{ij} be the primary and secondary arc costs of each arc $(i,j) \in A$. Furthermore, f_j denotes the cost of constructing a transshipment facility at location j.

We define a zero-one variable X_{ij} indicating whether a primary arc connects node i to node j and also another zero-one variable Y_{ij} indicating whether a secondary arc connects node i to node j or not. We also add one dummy node n+1 in the network G(N,A) such that the new network will be $G' = (N', A')$, where

$$N' = N \cup n+1,$$

$$A' = A \cup \bar{A}, \text{ and } \bar{A} = \{\text{arc } (i,n+1) \mid i \in N\}.$$

We define arc cost $C'_{i,n+1} = f_i$ for all $i \in N$. The problem can be formulated as follows :

$$\min \sum_{i=1}^{n-1} \sum_{j=2}^{n} C_{ij} X_{ij} + \sum_{i=1}^{n} \sum_{j=1}^{n+1} C'_{ij} Y_{ij} .$$

subject to the constraints :

$$\sum_{j=2}^{n} X_{1j} = 1 , \tag{5.8a}$$

$$\sum_{i=1}^{n-1} X_{in} = 1 , \tag{5.8b}$$

$$\sum_{i=1}^{n-1} X_{ij} - \sum_{k=2}^{n} x_{jk} = 0, \quad \text{for } j = 2,\ldots,n-1, \tag{5.8c}$$

$$\sum_{i \in Q} \sum_{j \in Q} X_{ij} \le |Q| - 1 \text{ for all } Q \subseteq N \text{ such that } |Q| \ge 2, \tag{5.8d}$$

$$\sum_{\substack{j=1 \\ j \ne i}}^{n+1} Y_{ij} = 1, \; i = 1,\ldots,n, \tag{5.8e}$$

$$\sum_{\substack{j=1 \\ j \ne i}}^{n+1} Z_{ij} - \sum_{\substack{k=1 \\ k \ne i}}^{n} Z_{ki} = 1, \; i = 1,\ldots,n, \tag{5.8f}$$

$$Z_{ij} - (n)\, Y_{ij} \le 0 \quad i = 1,\ldots,n,\ j = 1,\ldots,n,\ j \ne i, \tag{5.8g}$$

$$Y_{j,n+1} - \sum_{i=1}^{n-1} x_{ij} \le 0 \quad j = 2,\ldots,n-1, \tag{5.8h}$$

$$X_{ij} \in (0,1) \qquad \text{for all } (i,j) \in A\ , \tag{5.8i}$$

$$Z_{ij} \ge 0 \qquad \text{for all } (i,j) \in A', \tag{5.8j}$$

$$Y_{ij} \in (0,1) \qquad \text{for all } (i,j) \in A', \tag{5.8k}$$

where $$Y_{i,n+1} = \begin{cases} 1, & \text{if a transshipment facility is located at i} \\ 0, & \text{otherwise.} \end{cases}$$

Constraint sets (5.8a) - (5.8d) ensures that the solution will include a simple path of primary arcs from node 1 to node n. If dummy arcs $Y_{i,n+1}$ = 1, then a transshipment facility is located at node i; therefore, constraint (5.8h) requires the transshipment facility located on the primary path. Constraint sets (5.8e) - (5.8g), (5.8j) - (5.8k), ensures a spanning tree structure of secondary arcs on G′ rooted at node 1.

The problem can be solved by relaxing the constraint set (5.8h). We obtain Lagrangian subproblem as follows :

$$\text{(L)}\quad \min \sum_{i=1}^{n-1} \sum_{j=2}^{n} C_{ij} X_{ij} + \sum_{i=1}^{n} \sum_{j=1}^{n+1} C'_{ij} Y_{ij} + \sum_{j=1}^{n} \lambda_j \left(Y_{j,n+1} - \sum_{i=1}^{n-1} x_{ij}\right).$$

subject to the constraints (5.8a) - (5.8g),and (5.8i - 5.8k). The Lagrangian subproblem can be decomposed into the following two subproblems:

$$(\text{L}_1)\qquad \min \sum_{i=1}^{n-1} \sum_{j=2}^{n} (C_{ij} - \lambda_j)\, X_{ij}\ .$$

subject to the (5.8a) - (5.8d), (5.8i).

$$(\text{L}_2)\qquad \min \sum_{i=1}^{n} \sum_{j=1}^{n+1} \bar{C}_{ij}\, Y_{ij}\ .$$

subject to the (5.8e - 5.8g), (5.8j) - (5.8k), where

$$\bar{C}_{ij} = \begin{cases} C'_{ij} & \text{if } (i,j) \in A\ , \\ C'_{ij} + \lambda_j & \text{if } (i,j) \in A'. \end{cases}$$

Here $\lambda_j \geq 0$ is a Lagrangian multiplier.

The problem L_1 has the structure of a shortest path problem and the problem L_2 has the structure of a minimum spanning tree problem.

Application 5.9 : Design of mass transit system

Reference : Current and Pirkul (1991)

A state transport corporation wants to design the public mass transit system between two major cities. The primary mode might be a fixed rail system such as a subway, while the secondary mode might be a feeder system consisting of buses. Passenger stations as transshipment facilities are required at the locations on the primary subway where bus system will feed into subways. The problem of optimal design of mass transit system is also a two-level hierarchical network design problem like Application 5.8. We have to design the network such that there should be one primary rail system between two major cities and other cities should be connected either via a feeder bus system or should lie on the primary path.

Application 5.10 : Optimal design of oil/gas pipeline network

Reference : Current and Pirkul (1991)

Oil and gas commission of a country wants to connect a major oil/gas field with a transporting facility or a processing facility. Furthermore, in that geographical area several other smaller oil/gas fields are also present. The OGC wants to design a two-level hierarchical network of pipelines at minimum cost to transport all available oil/gas to the processing centre. Typically, a main pipe line connects the major oil/gas

field with the processing centre. Small oil/gas fields in the area are connected to the main line via secondary lines, which are called gathering lines and consist of pipes having much smaller diameters. The problem can also be formulated like Application 5.8.

Application 5.11 : Optimal transportation network design for resource extraction area

A big industrial house is planning to set-up a new processing centre to process Boxite for manufacturing aluminum. They are planning to set-up this plant near the boxite extraction area which consists of scattered boxite mines. The company wants to connect the processing centre to the national transport system, with a nearest site on the national highway to facilitate transportation of finished products. Furthermore, all mines in that geographical area have to be connected with processing centre to facilitate transport of raw material.

The problem is also a hierarchical network design problem with a primary path of railroad between processing centre and nearest site on national highway and each mine must either be on this primary path or be connected to primary path via a secondary transportation system (say trucks). Loading and unloading facilities are required at the interconnection of primary and secondary arcs. The problem will be similar to that in Application 5.8.

Application 5.12 : Optimal design of transportation network of a developing country

Reference : Current, Re-velle and Cohon (1986)

In a developing country, transportation network between two major cities has to be designed. The primary path could be thought as an all-weather highway. The secondary paths might be unimproved roads connecting intermediate villages to the all-weather highway. All intermediate village

allocated among various activities and b_k (for $k = 1,\ldots,K$) denotes the total availability of resource k. Let d_i denotes the duration of activity i. The problem can be formulated as follows :

$$\min \sum_t t\, Y_{nt}$$

subject to

$$\sum_t Y_{it} = 1, \quad i = 1,\ldots,n, \tag{5.14a}$$

$$\sum_t t(Y_{jt} - Y_{it}) \geq d_i \quad (i,j) \in A, \tag{5.14b}$$

$$\sum r_{ik} \left(\sum_{t-d_i+1}^{t} Y_{im} \right) \leq b_k, \qquad k = 1,\ldots,K, \quad t = 1,\ldots,T, \tag{5.14c}$$

$$Y_{it} \in \{0,1\}. \tag{5.14d}$$

Constraint (5.14a) ensures that every activity must start once. Constraint (5.14b) are the precedence constraint, where A is the set of all activities with precedence constraints and constraint (5.14c) are the resource constraints. Relaxing the resource constraints (5.14c) of the above formulation with Lagrangian multipliers $\lambda_{tk} \geq 0$, the relaxed problem will be :

$$\min \sum_t t\, Y_{nt} + \sum_i \sum_t \mu_{it} Y_{it} - \sum_k \sum_t \lambda_{tk} b_k.$$

subject to

$$\sum Y_{it} = 1, \qquad i = 1,\ldots,n,$$

$$\sum t(Y_{jt} - Y_{it}) \geq d_i \quad (i,j) \in A,$$

$$Y_{it} \in \{0,1\},$$

where

$$\mu_{it} = \sum_k r_{ik} \left(\sum_{m=t}^{t-d_i+1} \lambda_{mk} \right), \quad i = 1,\ldots,n; \; t = 1,\ldots,T.$$

The relaxed problem can be viewed as a generalization of a longest path computation in which we have to minimize not only the completion time but the sum of 'costs' μ_{it} of starting activity i at time t. Christofides (1987) suggested an algorithm to solve the relaxed problem.

Application 5.15 : Procurement of aviation fuels

Reference : Austin and Hogan (1976)

Defense department requires aviations fuels for several military installations. Generally, requirements for aviation fuels are met with purchases made in the usual competitive bidding environment. Invitation for bids are submitted to oil companies at frequent intervals annually. The objective of defense department is to procure the required fuel at the lowest laid-down (fuel plus transportation to demand sites) to the government. Generally, a wide variety of bidding options are available to the oil companies, such as availability of different transportation modes (pipe-line, tank car) or an oil company may indicate a total maximum offer which is less than the sum of maximum available at its shipping points. Furthermore, a company may "tie in" its bid to one or more separate companies, that is, it may make its maximum offer contingent upon the amount awarded to other bidders. Sometimes, companies may indicate a minimum acceptable quantity for one or more shipping points.

The problem has been formulated as a resource constrained minimum cost circulation problem by Austin and Hogan [1976]. Because of the features of "tie-in" between two companies, the total flow on some of the arcs must not exceed a limited quantity. This feature was dealt as a side constraint in the network. The additional side constraint was relaxed using Lagrangian relaxation technique and the resulting subproblem was a minimum cost circulation problem.

Application 5.16 : Optimal planning for a forest industry

Reference : Jörnsten and Väbrand (1976)

A forest industry is having various plants at different locations where at sawmills, timber logs are converted into timber and other by products such as pulp and paper in given proportions. There are several customers for these products and timber, which are having different but known demands for each of these. Furthermore, timber logs are supplied from different forests having known availability of timber logs. Timber can be either transported directly to the customers from the forests or to a plant where the timber is converted into pulp and paper. The cost of transportation of timber from different forests to customers or to the industries is known. Furthermore, transportation cost of pulp and paper from any plant to customers is also known.

The objective of optimal planning for a forest industry problem is to find out a minimum cost transportation plan of the timber and by-products, keeping in view the demands at various customer centres, supply of timber from different forests and conversion proportion of by-products at different plants.

The problem was formulated as a multicommodity generalized flow problem by [Jörnsten (1986)]. A Lagrangian relaxation of complicating conversion constraints at each plant resulted into a convex cost flow subproblem. For a complete study of the model, refer to Jörnsten and Värbrand (1986).

Application 5.17 : A logistics planning system

Reference : Klingman, Mote and Philips (1986)

A logistics planning system was developed for W.R. Grace company, one of the largest suppliers of phosphate-based chemical products of the USA by

[Klingman et al. (1986)]. The mathematical model underlying this system includes production, distribution policies of multicommodities in multiple time periods. They decompose the problem into a generalized network component and a small linear non-network component. The generalized network component was transformed into pure network and non-network constraints were relaxed by using Lagrangian relaxation technique. The relaxed problem was solved as a pure network flow problem. As the model is tidious and complex, it is difficult to present it in our report, we decided to omit the description of the model. For a complete study of the model, refer to Klingman et al. [1986].

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